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The Smallest Non-Hamiltonian 3-Connected Cubic Planar Graphs Have 38 Vertices

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We show that all 3-connected cubic planar graphs on 36 or fewer vertices are hamiltonian, thus extending results of Lederberg, Butler, Goodey, Wegner, Okamura, and Barnette. Furthermore, the only non-hamiltonian examples on 38 vertices which are not cyclically 4-connected are the six graphs which have been found by Lederberg, Barnette, and Bosák.

1. INTRODUCTION

Throughout this paper, a *C3CP* is a cubic 3-connected planar graph, and G is any non-hamiltonian C3CP of least order. Define $n = |VG|$. Then, successively, Lederberg [12] ($n \geqslant 20$), Butler [5] and Goodey [8] ($n \geqslant 24$), Barnette and Wegner [2] ($n \geqslant 28$), and Okamura [15, 16] ($n \geqslant 34$) have established lower bounds on n. Various non-hamiltonian C3CPs on 38 vertices have been constructed by Lederberg, Barnette, and Bosák [4]. These are shown in Fig. 1.1.

In this paper we extend the method of Okamura to demonstrate that $n = 38$. Furthermore, the only non-hamiltonian C3CPs on 38 vertices with non-trivial 3-cuts are those shown in Fig. 1.1. We also discuss non-hamiltonian C3CPs satisfying stronger connectivity conditions, in particular those which are 4- or 5-cyclically connected.

Before proceeding we need some definitions. By a *k-gon* we mean a face of a planar graph bounded by k edges. Note that a k-cycle is not

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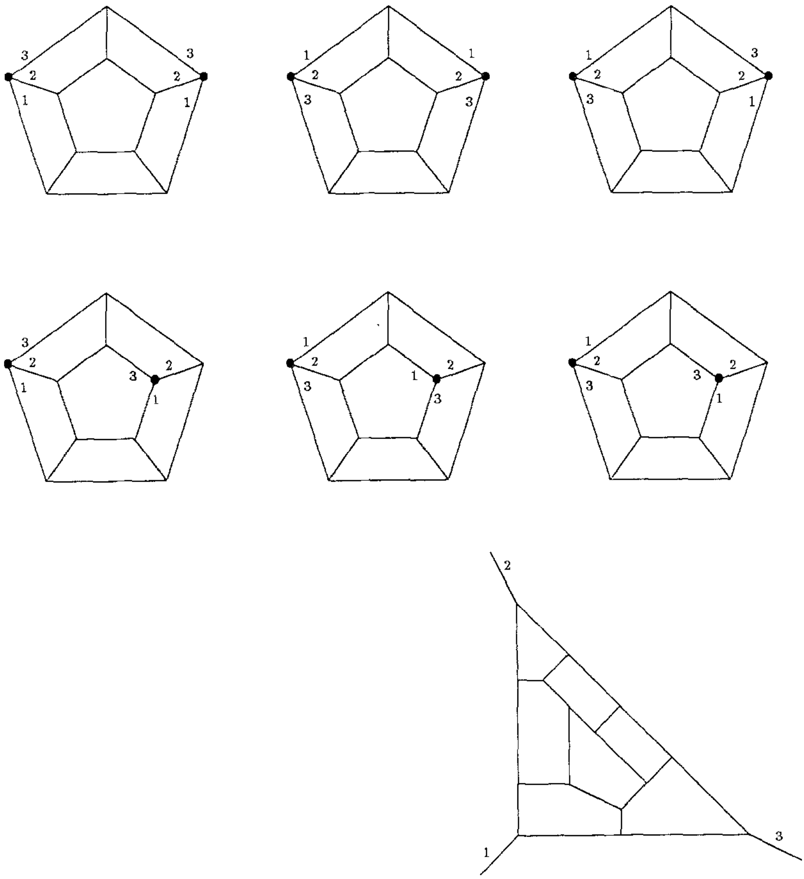

FIG. 1.1. In each of the diagrams, replace the dark vertices by the 3-piece on the right.

necessarily a k-gon. By a *k-cut* we mean a set of k edges whose removal leaves the graph disconected and of which no subset has that property. The two components formed by removal of a k-cut are called *k-pieces.* A k-cut is *non-trivial* if each of its k-pieces contains a cycle and *essential* if it is non-trivial and each of its k-pieces contains more than k vertices. It is *non-essential* if it is non-trivial and not essential. A cubic graph is *cyclically k-connected* if it has no non-trivial t-cuts for $0 \leqslant t \leqslant k-1$, and *exactly cyclically k-connected* if in addition it has at least one non-trivial k-cut.

We can now state our main results. The proofs can be found near the end of Section 3.

THEOREM 1.1. *Every C3CP with* 36 *or fewer vertices is hamiltonian.*

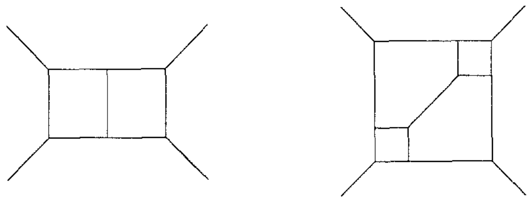

FIGURE 1.2

THEOREM 1.2. *Let H be a non-hamiltonian C3CP with* 38, 40, *or* 42 *vertices. Then one of the following is true.*

(a) *H is one of the six C3CPs on* 38 *vertices with* 3-*cuts shown in Fig.* 1.1.

(b) *H has* 40 *or* 42 *vertices and has at least one* 3-*cut.*

(c) *H has* 42 *vertices, is cyclically* 4-*connected, and has an essential* 4-*cut. Furthermore, for one such* 4-*cut, one of the* 4-*pieces is the first one shown in Fig.* 1.2 *and the other is obtainable from a cyclically* 4-*connected non-hamiltonian C3CP on* 38 *vertices by the inverse of one of the operations shown in Fig.* 1.3.

(d) *H is exactly cyclically* 4-*connected and has no essential* 4-*cuts.*

Our method of proof is similar to that used by Okamura [16]. Faulkner and Younger [7] have established that G is not cyclically 5-connected. In Section 2 we employ a variety of decomposition techniques, and some computation, to show that G does not have 3-cuts or essential 4-cuts. In Section 3 we prove that any remaining possibilities for G with $n \leqslant 36$ could be reduced to a smaller non-hamiltonian C3CP by applying one of Okamura's 15 reductions.

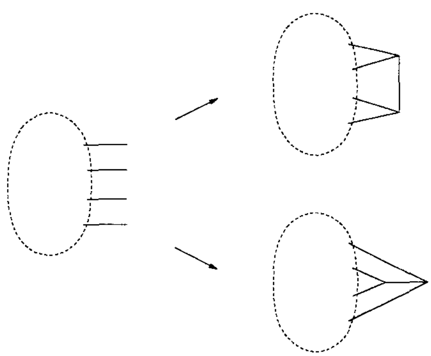

FIGURE 1.3

We will find the following lemmas very useful for the elimination of many subcases.

LEMMA 1.3. *Let the faces of a connected cubic planar graph be of size* $k_1, k_2, ..., k_r$.

(a) *It is not possible that exactly one of* $k_1, k_2, ..., k_r$ *be not divisible by* 5.

(b) *If exactly two of* $k_1, k_2, ..., k_r$ *are not divisible by* 5 *then those two faces are not adjacent.*

Proof. See page 272 of Grünbaum [9]. ∎

LEMMA 1.4. *Let H be a cyclically* 4-*connected C3CP with no essential* 4-*cut. Suppose that F is a k-gon of H, and let* $f_1, f_2, ..., f_k$ *be the faces other than F adjacent to each of the edges of F, in cyclic order. If no other face of H is a* 4-*gon, then*

$$|VH| \geqslant \sum_{i=1}^{k} (2f_i + \max(f_i - 5, 0)) - 6k.$$

Proof. Note that in the statement of the lemma we use f_i to denote both a face and the size of that face. We will adopt this convention throughout the paper.

The faces $f_1, f_2, ..., f_k$ are distinct since otherwise H is not 3-connected. Since H is cyclically 4-connected, it has no 3-gons.

Define $l = \sum_{i=1}^{k} f_i - 4k$. Let $g_1, g_2, ..., g_l$ be the faces adjacent to the outside boundary of $f_1, f_2, ..., f_k$, in cyclic order. These faces are distinct, since H has no essential 4-cuts.

For $f_i \geqslant 6$, let $g_{j_i}, g_{j_i+1}, ..., g_{j_i+m_i}$, where $m_i = f_i - 5$, be the faces adjacent to f_i and to no other $f_{i'}$. By assumption, $g_j \geqslant 5$ for $j_i \leqslant j \leqslant j_i + m_i$ so, by the connectivity of H, there is at least one vertex in g_j which is no other $g_{j'}$. Hence

$$\begin{aligned} |VH| &\geqslant 2k + 2l + \sum_{i=1}^{k} \max(m_i, 0) \\ &= \sum_{i=1}^{k} (2f_i + \max(f_i - 5, 0)) - 6k. \quad \blacksquare \end{aligned}$$

2. COMPUTATIONAL RESULTS

In this section we describe the computations which form the initial foundations of our investigation. Essentially, they enable us to restrict our

TABLE I

Counts of Subclasses of TFC3CPs

n	n_3	n_{e4}	n_4	n_5	n_a	n_b	n_A	Total
8	—	—	1	—	—	—	—	1
10	—	—	1	—	—	—	—	1
12	—	2	—	—	—	—	—	2
14	1	3	1	—	—	—	—	5
16	2	8	2	—	1	—	—	12
18	9	22	3	—	1	—	—	34
20	43	77	9	1	4	—	—	130
22	212	285	28	—	13	—	—	525
24	1115	1259	97	1	58	1	6	2472
26	6156	5863	378	1	279	7	27	12400
28	34693	29322	1601	3	1406	26	167	65619
30	199076	151308	7116	4	7525	146	967	357504

Note. n_3, with 3-cuts; n_{e4}, with essential 4-cuts but no 3-cuts; n_4, with no essential 4-cuts or 3-cuts; n_5, cyclically 5-connected; n_a, with at least one a-edge; n_b, with at least one b-edge; n_A, with at least one A-edge.

attention to C3CPs without 3-cuts or essential 4-cuts and provide us with a complete list of small C3CPs with certain exceptional edges. We also take the opportunity to investigate non-hamiltonian C3CPs with essential 4-cuts, but no 3-cuts, for $n \leqslant 42$.

A TFC3CP is a C3CP without 3-gons. Our major computation was the generation of all TFC3CPs with up to 30 vertices and a certain subset of those on 32 vertices. The method used was that of Mohar [14], in conjuction with the graph isomorphism system described by McKay [13]. The numbers of TFC3CPs found, under isomorphism as abstract graphs, are summarized in Table I.

Following Bosák [3], an *a-edge* is an edge which is present in every hamiltonian cycle, while a *b-edge* is absent from every hamiltonian cycle. We further define an *A-edge* to be an a-edge x in C3CP H whose image x is an a-edge in Flip(x, H). The latter is defined in Fig. 2.1.

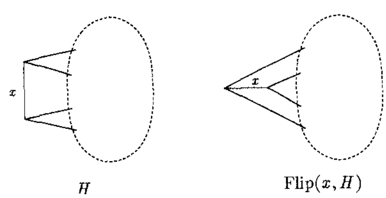

FIGURE 2.1

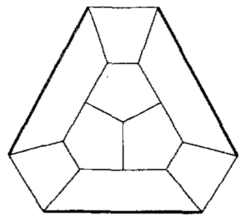

FIGURE 2.2

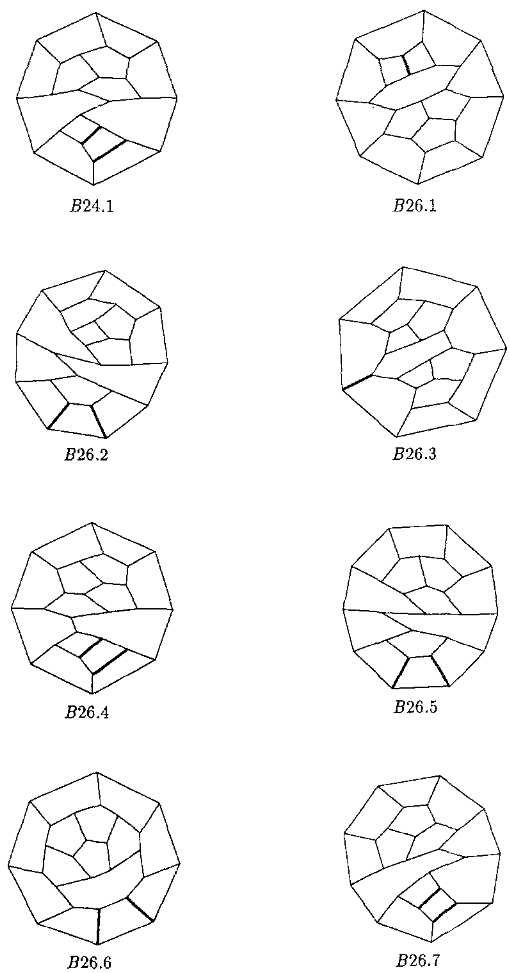

FIGURE 2.3

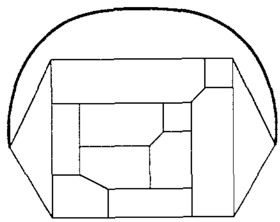

FIGURE 2.4

The unique TFC3CP on 16 vertices with a-edges is shown in Fig. 2.2. The eight TFC3CPs on 24 or 26 vertices with b-edges are shown in Fig. 2.3. One of the six TFC3CPs on 24 vertices with an A-edge is shown in Fig. 2.4. In each case the edges with the required property are those drawn bold.

We now consider non-hamiltonian C3CPs with 3-cuts. It was shown by Butler [6] that, if any minimal non-hamiltonian C3CP H has a 3-cut, then it has 38 vertices. The principal technique used by Butler was to separate H into two smaller C3CPs at the 3-cut, as shown in Fig. 2.5.

Our computations enable us to prove the following somewhat stronger theorem.

THEOREM 2.1. *Let H be a non-hamiltonian C3CP with a* 3*-cut and at most* 38 *vertices. Separate H into two parts as in Fig.* 2.5. *Then either H_1 or H_2 is non-hamiltonian, or H is one of the six non-hamiltonian C3CPs on* 38 *vertices shown in Fig.* 1.1.

Proof. Suppose that H_1 and H_2 are hamiltonian. Then, as in [6], one of the pairs $\{x', x''\}$, $\{y', y''\}$, and $\{z', z''\}$ consists of an a-edge and a

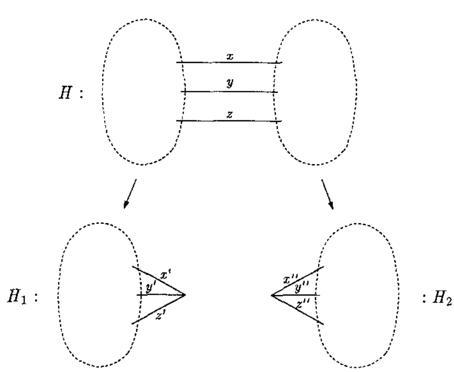

FIGURE 2.5

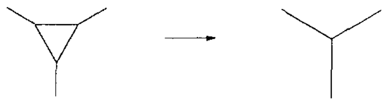

FIGURE 2.6

b-edge. It is clear that a minimal C3CP with an a-edge cannot contain a 3-gon, since otherwise the reduction shown in Fig. 2.6 would produce a smaller C3CP with an a-edge.

Similarly, a minimal C3CP with a b-edge cannot contain a 3-gon. It follows from Table I that in each case the minimal C3CPs are unique and are those shown in Figs. 2.2 and 2.3. Joining them together in every possible manner, we find the six non-isomorphic C3CPs of Fig. 1.1. We note that these examples were first found by Lederberg, Barnette, and Bosák, and that the representation shown in Fig. 1.1 is due to Bosák [4]. ∎

We now turn to cyclically 4-connected C3CPs with essential 4-cuts. Following Butler [6], we can separate such a graph at an essential 4-cut into two 4-pieces and reassemble these into cubic graphs as in Fig. 2.7. The following lemma is proved in [6].

LEMMA 2.1. *Suppose L, L', R, and R' are hamiltonian but H is not hamiltonian. Then*

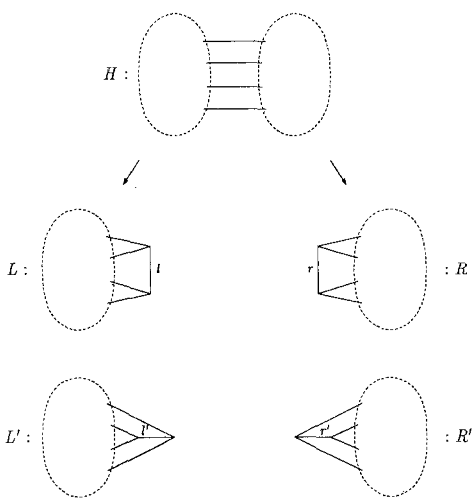

FIGURE 2.7

(a) *at least one of l and r', and one of l' and r, is an a-edge, and*

(b) *at least one of L and L', and one of R and R', is cyclically 4-connected.* ∎

Lemma 2.1 enables us to greatly simplify the search for a non-hamiltonian cyclically 4-connected C3CP H with an essential 4-cut. Choose the essential 4-cut to minimize $|VL|$. Then, if $|VG| \leqslant 44$, there are three possibilities:

(1) Either R or R' is non-hamiltonian.

(2) One of l and l' is an a-edge and one of L and L' is cyclically 4-connected. Similarly one of r and r' is an a-edge and one of R and R' is cyclically 4-connected.

(3) r and r' are A-edges and one of R and R' is cyclically 4-connected.

As stated earlier, we have generated all TFC3CPs with at most 30 vertices. By removing appropriate edges from them, we have found all possible 4-pieces of the form required for possibility (2) to 28 vertices and all of the possible right 4-pieces required for possibility (3) to 28 vertices. All of the latter possible 4-pieces on 30 vertices were also found, by generating just those 32-vertex TFC3CPs which were needed. By joining together 4-pieces in the manner required for possibilities (2) and (3), we obtained the following theorem.

THEOREM 2.2. *Let H be a non-hamiltonian C3CP which is cyclically 4-connected but has an essential 4-cut. Separate H into two 4-pieces P_1 and P_2 at an essential 4-cut so that $|VP_1|$ is minimized. If $|VH| \leqslant 42$ then one of the following is true.*

(a) *P_1 is one of the two 4-pieces of Fig. 1.2, and one of the C3CPs formed from P_2 as shown in Fig. 1.3 is non-hamiltonian.*

(b) *H is one of the two non-hamiltonian C3CPs on 42 vertices shown in Fig. 2.8.*

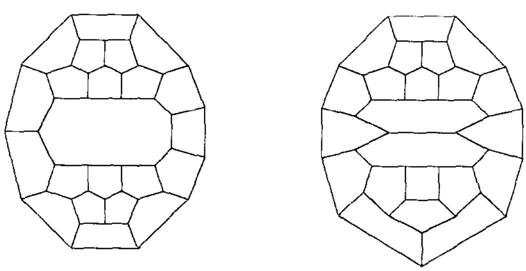

FIGURE 2.8

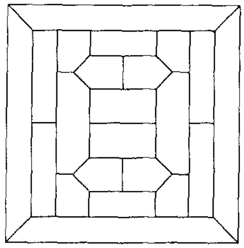

FIGURE 2.9

Proof. The only possibilities for P_1 which have 10 or fewer vertices are those shown in Fig. 1.2. All other small 4-pieces are either not minimal or have 3-cuts which necessarily are also 3-cuts in any C3CP formed from them by joining with another 4-piece. Furthermore, if P_1 is one of these two, and each of the two C3CPs formable from P_2 as in Fig. 1.3 have hamiltonian cycles, then at least one of those cycles can be extended to a hamiltonian cycle in H.

If $|VP_1| \geqslant 12$, then $|VP_2| \leqslant 30$. All the possibilities are then within the limits of our computations. The only non-hamiltonian C3CPs found either had 3-cuts or were isomorphic to one of those shown in Fig. 2.8. ∎

The first graph in Fig. 2.8 was found by Faulkner and Younger [7]. The second is new. We should note here that [7] appears to describe a computer search which should have found both the graphs in Fig. 2.8. However, a more careful reading of [7] indicates that the search on 42 vertices was not intended to be complete.

The only other known non-hamiltonian cyclically 4-connected C3CP on 42 or fewer vertices was found by Grünbaum [9] and appears in Fig. 2.9. It has 42 vertices and only non-essential 4-cuts.

The smallest known non-hamiltonian cyclically 5-connected C3CP has 44 vertices and appears in Fig. 2.10. It is due to Tutte [10]. The minimality has been established by Faulkner and Younger [7], but the uniqueness remains open.

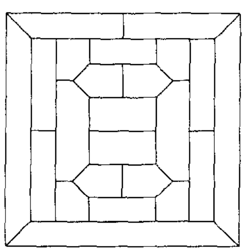

FIGURE 2.10

3. Proofs of the Main Results

We give a sequence of lemmas to facilitate the proofs of Theorems 1.1 and 1.2. Many details of the proofs have been omitted in the interests of space. A reader interested in the whole story can find it in [11].

Throughout this section G is a minimal non-hamiltonian C3CP with 36 or fewer vertices. From Theorems 2.1 and 2.2, we know that G has no 3-cuts or essential 4-cuts, and from [7] we know that G is not cyclically 5-connected.

LEMMA 3.1. *G cannot contain adjacent 4-gons.*

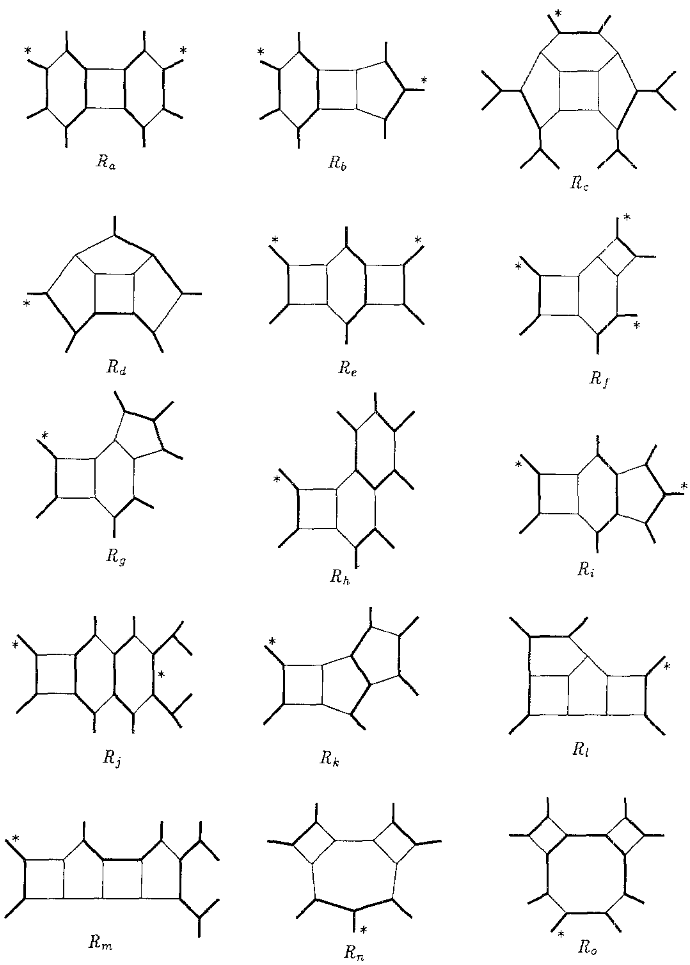

FIGURE 3.1

LEMMA 3.2. *G cannot contain a k-piece as illustrated in Fig.* 3.1.

Proof. The proof is essentially that of Okamura [16] except where the asterisked edges correspond to b-edges of the reduced graph G', and G' is either one of the graphs of Fig. 2.3 or the graph B24.1 with one vertex expanded to a 3-gon. This gives a few hundred exceptional cases which can be examined separately. ∎

COROLLARY 3.3. *Each* 7*-gon or* 8*-gon of G is adjacent to at most two* 4*-gons.*

COROLLARY 3.4. *Each* 4*-gon of G is adjacent to at least two k-gons with* $k \geqslant 7$.

LEMMA 3.5. *Let G be a minimal non-hamiltonian* 3*-connected cubic planar graph. Let R be a cycle in G which contains at least five faces in its interior. Then if there is a* 4*-gon in the interior of R there is at least one k-gon, for* $k \geqslant 6$, *in the interior of R.*

Proof. Suppose the interior of R contains no k-gon for $k \geqslant 6$. By Corollary 3.4 we have the three configurations of Fig. 3.2. By Lemma 3.2(k), a, b, c, and c' must all be 4-gons. Then Fig. 3.2(iii) contradicts Lemma 3.2(m).

If $a = 4$, then the interior of R contains only four faces, in contradiction to the hypothesis of the lemma. If $b = 4$, then it must be adjacent to a 4-gon or a 5-gon, but this contradicts Lemma 3.1 and Lemma 3.2(m). Hence the lemma follows. ∎

LEMMA 3.6. *Let G be a minimal non-hamiltonian* 3*-connected cubic planar graph. Let R be a cycle in G which contains at least five faces in its interior. If R contains at least one* 4*-gon and exactly one k-gon, for* $k \geqslant 6$, *in its interior, then G contains one of the configurations of Fig.* 3.3.

Proof. This result follows via a similar argument. ∎

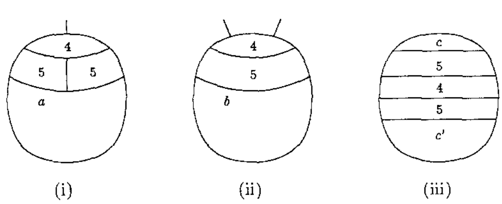

FIGURE 3.2

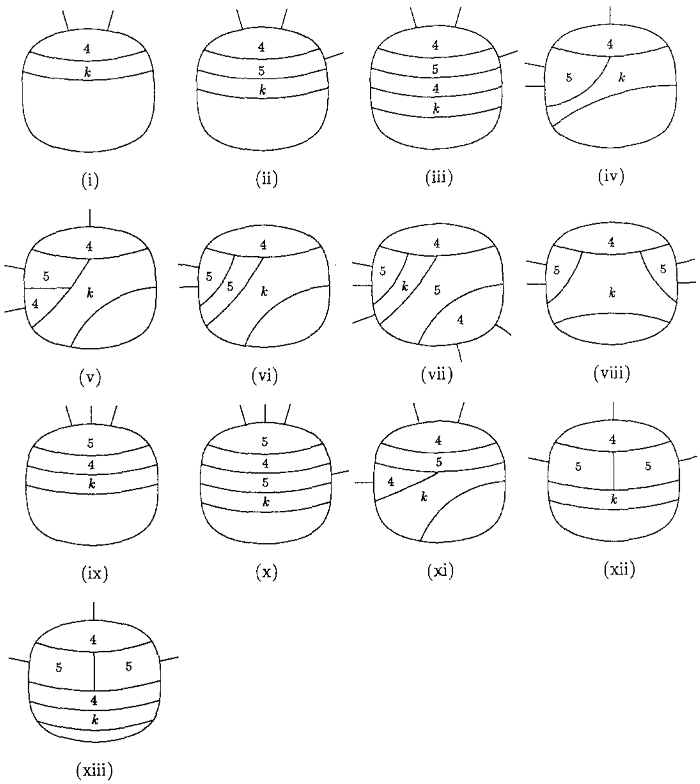

FIGURE 3.3

We now do some elementary counting. If p_k is the number of k-gons of G, then the Euler polyhedral formula yields

$$2p_4 + p_5 = 12 + \sum_{k \geqslant 7} (k-6)\, p_k. \tag{1}$$

Further,

$$\sum_{k \geqslant 4} = \begin{cases} 19, & \text{for} \quad n = 34, \\ 20, & \text{for} \quad n = 36. \end{cases} \tag{2}$$

Combining (1) and (2) gives

$$p_4 = p_6 + \sum_{k \geqslant 7} (k-5)\, p_k - \begin{cases} 7, & \text{for} \quad n = 34, \\ 8, & \text{for} \quad n = 36. \end{cases} \tag{3}$$

By Lemma 3.1, every k-gon for $k \geqslant 9$ is adjacent to at most $\lfloor k/2 \rfloor$ 4-gons. Corollary 3.3 and 3.4 then give

$$2p_4 \leqslant 2p_7 + 2p_8 + \sum_{k \geqslant 9} \lfloor k/2 \rfloor p_k. \tag{4}$$

Combining (3) and (4) gives

$$p_6 + p_7 + \sum_{k \geqslant 8} 2p_k \leqslant \begin{cases} 7, & \text{for } n = 34, \\ 8, & \text{for } n = 36. \end{cases} \tag{5}$$

LEMMA 3.7. *G contains no* 4-*gon adjacent to a* 6-*gon.*

Proof. The techniques are again those of the corresponding result in [16]. There are many more cases to consider here and it is often useful to employ Lemma 3.5 or 3.6. ∎

LEMMA 3.8. *G contains a* 4-*gon adjacent to a* 5-*gon.*

THEOREM 3.9. *All* 3-*connected cubic planar graphs of order* 34 *or* 36 *are hamiltonian.*

Proof. The proof here corresponds to that of Theorem 1 in [16] but there are many more cases to be dealt with. Those cases which are not straightforward are dealt with by Lemmas 1.3, 1.4, 3.5, or 3.6. ∎

Proof of Theorem 1.1. Suppose G is a minimal non-hamiltonian C3CP with 36 or fewer vertices. By Okamura [16], $|VG| \geqslant 34$. G is cyclically 4-connected by Theorem 2.1 and has no essential 4-cuts by Theorem 2.2. It is not cyclically 5-connected by Faulkner and Younger [7]. The non-existence of G now follows from Theorem 3.9. ∎

Proof of Theorem 1.2. This follows from Theorems 2.1, 2.2, and 1.1. In part (c), the use of the C3CPs of Fig. 2.6 is excluded by the fact that they each have two disjoint 3-cuts, one of which must remain in H. ∎

Finally, we note some problems which this paper does not solve.

(a) What is the smallest size of a cyclically 4-connected non-hamiltonian C3CP? Three examples are known on 42 vertices (Figs. 2.8 and 2.9) but the possibilities 38 and 40 remain open.

(b) Is the minimal (44 vertex) non-hamiltonian cyclically 5-connected C3CP of Fig. 2.10 unique? This question can probably be answered by direct computation.

ACKNOWLEDGMENT

We thank Joan McKay for the considerable effort required to compose the more than 50 figures required for this paper and the technical report version [11].

Note Added in Proof. A recent paper of Barnette [1] demonstrates that there is no non-Hamiltonian C3CP on 34 vertices. The methods used are not dissimilar to our own.

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